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SECTION THREE

Convex Sets in Linear Topological Spaces

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1. Introduction

IN this paper we shall prove some theorems concerning convex sets in linear topological spaces.

In §2 we shall show that if K and K_1 are non-empty, disjoint convex sets in a real-linear topological space and if at least one of them has inner points then there is a hyperplane separating them. This was proved by M. Eidelheit [3] for the more special case that the space is linear and normed and both K and K_1 have inner points.

From this result we deduce that if K is a convex set with inner points then through every boundary point of K there passes a supporting hyperplane; more generally if R is a linear set containing no inner points of K then there is a hyperplane H containing R such that K lies on one side of H. This was proved for linear normed spaces by S. Mazur [4], the first half of this result having been obtained previously by G. Ascoli [1] for separable, linear, normed spaces.

Aside from greater generality, our proofs are more elementary than those of Ascoli, Mazur, and Eidelheit and do not use Minkowski functionals or the Hahn-Banach extension theorem. (The simplified proofs for Eidelheit's theorem due to Kakutani [5] and Botts [2] require the theorem of Ascoli-Mazur.)

In §3, we consider linear topological spaces with either the real or complex numbers as scalars. We repeat the argument of Ascoli and Mazur to obtain the theorems (proved by them for the real-linear normed case) that every closed convex set is weakly closed and that if $x_n \to x$ weakly then x is in the closure of the set of all averages $c_1x_1 + \ldots + c_mx_m$ with $m = 1, 2, \ldots$, all $c_n \geqslant 0$ and the sum of the c_n equal to 1; however, for this result we require the stronger restriction on the space that every open set containing the origin should contain a convex set having the origin as inner point.

2. Separating Hyperplanes

Let E be a linear space topologized by a family of open sets such that: the set-union of an arbitrary collection (possibly empty) of open sets is an open set, the set-intersection of two open sets is an open set, and each point of E is the intersection of all open sets containing it. We shall suppose (i) $x + y$ is continuous in x for fixed y and (ii) cx is continuous in the pair c, x. It follows that whenever 0 is in the closure of a set of x then 0 is also in the closure of the cx where c may depend on x providing that $K = \sup|c|$ is finite; indeed, for arbitrary integer p each cx may be expressed as $m2^{-p}x + r_p x$ with $0 \leqslant r_p < 2^{-p}$ and m one of the finite set of integers numerically less than $2^p(2 + K)$, and hence by choosing p sufficiently large it can be seen that in every open set containing 0 there is at least one cx.

If S_1, S_2 are subsets (possibly single points) of E then $S_1 + S_2$ shall denote the set of all $x_1 + x_2$ with x_1 in S_1 and x_2 in S_2, and cS the set of all cx with x in S; $J(S_1, S_2)$, the *join* of S_1 and S_2, shall denote the set of all $cx_1 + (1 - c)x_2$ with x_1 in S_1, x_2 in S_2, and $0 \leqslant c \leqslant 1$. Our conditions on E imply that $x + N$ and cN are open sets for fixed x and $c \neq 0$ respectively if N is an open set.

A set K is called convex if $K = J(K, K)$. It is easily verified that $J(S_1, S_2)$ is convex if S_1, S_2 are both convex and that the set union of an increasing family of convex sets is again convex.

Our conditions on E ensure that if K is convex then its closure $\bar{K}$ is also convex; for if $K \subset K_1$ for some convex $K_1 \subset \bar{K}$ and if x is in $\bar{K}$ then $J(x, K_1)$ is again convex and $K \subset K_1 \subset J(x, K_1) \subset \bar{K}$ and it follows that K is contained in a maximal such K_1 which must then coincide with $\bar{K}$.

For the rest of this section we shall suppose that the scalars are the real numbers.

$l(x, y)$ shall denote the set of all $tx + (1 - t)y$ with $-\infty < t < \infty$; $l(x)$ shall mean $l(0, x)$. A set H is called linear if H contains $l(x, y)$ for all x, y in H; H is called a *hyperplane* if it is closed, linear, and different from E and the only linear sets which contain H are H and E.

Suppose H is a hyperplane. Choose any x_0 in H, any y_0 not in H and set $u = y_0 - x_0$. Then the linear set $H + l(u)$ must coincide with E and hence every x in E can be represented in the form $h + tu$ with h in H. Clearly h and t are determined uniquely by x. Let 0 have the representation $h_0 + t_0 u$ and for every $x = h + tu$ set $f(x) = t - t_0$. Then $f(x)$ is a real-valued linear functional on E and our conditions on E together with the closure of H ensure that $f(x)$ is continuous. (If $f(x)$ were not continuous we could deduce that for some $\epsilon > 0$ and

some h_1, t_1 and some set of h, t, $|t - t_1| > \epsilon$ but $h_1 + t_1 u$ is in the closure of $h + tu$; hence 0 is in the closure of the set of $h - h_1 + (t - t_1)u$; then, since $|t - t_1|^{-1} < \epsilon^{-1}$, 0 is in the closure of the set of $(t - t_1)^{-1}(h - h_1) + y_0 - x_0$ and finally y_0 is in the closure of H.) H consists precisely of the x for which $f(x) = -t_0$. Each of the sets of x for which $f(x) \leqslant -t_0$, and $f(x) \geqslant -t_0$ respectively, is called *a side of H* and also a *half-space.* It is easily verified that the two sides of H are uniquely determined by H and that if y is an inner point of a set S lying on one side of H then y is not in H.

Conversely, if H is a closed linear set with $H \neq E = H + l(u)$ for some u then it is easily verified that H is a hyperplane. Also, if $f(x)$ is any real-valued continuous linear functional not identically zero and c is a real number then the set of x for which $f(x) = c$ is a hyperplane.

We shall say that two sets S_1 and S_2 are separated by the hyperplane H if they lie on different sides of H, equivalently if, for some continuous real-valued linear functional $f(x)$ and some c, $f(x) = c$ if and only if x is in H, $f(x) \leqslant c$ for all x in S_1 and $f(x) \geqslant c$ for all x in S_2.

A hyperplane H is called a *supporting* hyperplane of a set S if S lies on one side of H and H contains at least one boundary point of S.

LEMMA. *If C is a closed set with inner points and its boundary H is a non-empty linear set then H is a hyperplane and C is one of its sides.*

Proof. (i) By a translation, if necessary, we may suppose that the origin 0 is in H.

(ii) Since C is closed and its boundary is a linear set it follows that C is convex. Hence if u is in C then tu is in C for all $t \geqslant 0$ and $h + tu = (1 + t)\,((1 + t)^{-1} h + t(1 + t)^{-1}u)$ is in C for all h in H and $t \geqslant 0$.

(iii) If u and $-u$ are both in C then u is in H. For otherwise there would be an open set N containing u and contained in C. It would follow that $J(N, -u)$ is contained in C and hence $\frac{1}{2}(N + (-u))$, which is an open set containing the origin, is contained in C, contradicting the fact that the origin is a boundary point of C.

(iv) If u_0 is a fixed point in C but not in H then C consists precisely of all $h + tu_0$ with h in H and $t \geqslant 0$. Indeed, if v in C were not of this form it would follow that for every $t \geqslant 0$, $v - tu_0$ is not in H. Since v would be an inner point of C it would follow that for every $t > 0$, $v - tu_0$ is in C, $t^{-1}(v - tu_0) = t^{-1}v - u_0$ is in C. When $t \to \infty$ we obtain $-u_0$ is in C, assumed closed. By (iii) it would follow that u_0 is in H contradicting our assumption.

(v) With the u_0 of (iv), $H + l(u_0) = E$ since it is a linear set having inner points. Since H is closed this implies the lemma.

THEOREM 1. *Suppose that K and K_1 are non-empty disjoint convex sets and that K is open. Then there is a continuous linear $f(x)$ and a c such that $f(x) < c$ for all x in K and $f(x) \geqslant c$ for all x in K_1, so that the hyperplane $f(x) = c$ separates K and K_1.*

Proof. (i) Since the union of an increasing family of convex sets, each disjoint from K, is also a convex set disjoint from K we may suppose that K_1 is not contained in any other convex set disjoint from K. Then K_1 will be closed since the closure of K_1 is convex and disjoint from the open set K.

(ii) If u is not in K_1 then $J(u, K_1) \neq K_1$. But the set $J(u, K_1)$ is convex and contains K_1 so it must intersect K. Thus $cu + (1 - c)x_1$ must be in K for some x_1 in K_1 and some $0 < c \leqslant 1$.

(iii) If x_1 is in K_1 and u is arbitrary then at least one of the sets: all $x_1 + tu$, $t \geqslant 0$ *or* all $x_1 - tu$, $t \geqslant 0$ is contained in K_1. Otherwise there would exist u_1 and u_2, neither in K_1 and with x_1 in $J(u_1, u_2)$. Then by (ii) there would be y_1, z_1 with z_1 in K_1, y_1 in K and y_1 contained in $J(u_1, z_1)$; similarly for u_2, y_2, z_2. Then $J(y_1, y_2)$ would intersect the triangle of x_1, z_1 and z_2, that is, $J(x_1, J(z_1, z_2))$, in a point w which would be in both K and K_1, a contradiction.

(iv) If x_1 is in K_1 but u is not in K_1 then for every $t > 0$, $x_1 + t(x_1 - u)$ is an inner point of K_1. For there is an open set N containing u and disjoint from K_1. Then $x_1 + t(x_1 - y)$ is in K_1 for all $t \geqslant 0$ and all y in N, and for fixed $t > 0$, $x_1 + t(x_1 + (-1)N)$ is an open set containing $x_1 + t(x_1 - u)$.

(v) Since K_1 is a convex set it now follows that the boundary of K_1 is a linear set and that K_1 is a closed set with inner points. The theorem then follows from the previous lemma.

COROLLARY 1. *If K and K_1 are non-empty convex sets and K has inner points, then K and K_1 can be separated by a hyperplane whenever K_1 is contained in the closure of a convex set K' such that K' contains no inner point of K, in particular if K_1 contains inner points but K and K_1 have no common inner points.*

Proof. This follows from the fact that if a convex set has inner points it is contained in the closure of the convex set of its inner points.

COROLLARY 2. *If x is a boundary point of a convex set K possessing inner points then there exists a supporting hyperplane of K which contains x.*

Proof. This follows from Corollary 1 if x is taken as K_1.

THEOREM 2. *Suppose that K is a convex set with inner points and*

that R is a non-empty linear set containing no inner points of K. Then R is contained in a hyperplane H such that K lies on one side of H.

Proof. Replacing K by the convex set of its inner points, we may assume that K is an open set and that K and R are disjoint.

Set $J = J(R, K)$. Then J is a convex set with inner points. Moreover no inner point of J is in R. Indeed, if x in R were an inner point of J we could choose any x_0 in K and deduce that for some $t > 0$, $x + t(x - x_0) = y_1$ is in J but not in R and x is in $J(x_0, y_1)$. Then y_1 would be in $J(x_1, y)$ for some x_1 in K and y in R. This would imply that $l(x, y)$, all in R, intersects $J(x_0, x_1)$ all in K, contradicting the assumption that K and R are disjoint.

Corollary 1 to Theorem 1 now implies that J and R can be separated by a hyperplane H. Since R is contained in J, R must lie on both sides of H and hence is contained in H.

THEOREM 3. *If E is such that every open set containing 0 contains a convex subset with 0 as an inner point, then if K is a non-empty closed convex set and y is not in K there is an open set N containing y such that K and N can be separated by a hyperplane H. (Necessarily, y is not in H.)*

Proof. By our assumptions on E there will be an open set N containing y with N contained in a convex set K_1 such that K and K_1 are disjoint. Since K_1 has inner points it follows from Corollary 1 to Theorem 1 that K and K_1, *a fortiori* K and N, can be separated by a hyperplane.

COROLLARY. *For E as in Theorem 3, every closed convex set different from E is the intersection of the half-spaces which contain it.*

3. WEAK CONVERGENCE

If E is a linear space with a topology as specified in §2 we shall say that a sequence x_n in E converges *weakly* to x if $f(x_n)$ converges to $f(x)$ for every continuous linear $f(x)$ on E which is real or complex valued according as the scalars for E are the real or complex numbers. A set Z contained in E is called *weakly* closed if x is in Z whenever a sequence x_n in Z converges weakly to x.

If the scalars are the complex numbers then E may be regarded also as a linear space E' with the real numbers as scalars and with the same open sets as E. If $f(x)$ is complex-valued and linear and continuous on E, then clearly $\phi(x) = \mathscr{R}f(x)$ is real-valued, linear and continuous on E' and conversely for every real-valued, linear, continuous $\phi(x)$ on E', $f(x) = \phi(x) - i\phi(ix)$ is complex-valued, linear and continuous on E.

THEOREM 4. *If every open set containing the origin has a convex subset with the origin as inner point, then:*

(i) *Every closed convex set is weakly closed.*

(ii) *If x_n converges weakly to x then x is in the closure of the set of all averages $c_1x_1 + \ldots + c_mx_m$ with $m = 1, 2, \ldots$, all $c_n \geqslant 0$ and the sum of the c_n equal to 1. In particular, if E satisfies the first countability axiom of Hausdorff there will be a sequence of such averages which converge to x in the sense of the open-set topology.*

Proof. We may suppose that the scalars are the real numbers since x_n converges weakly to x for complex scalars if and only if this is true for real scalars.

If now K is a closed convex set and x_n in K converges weakly to y and y is not in K then Theorem 3 implies that there is a hyperplane H separating K and y with y not in H. Then there is a continuous real-valued linear $f(x)$ and a real number c such that $f(x) \leqslant c$ for all x in K but $f(y) > c$. This contradicts the fact that $f(x_n)$ must converge to $f(y)$.

(After this note was accepted for publication the writer learned that an elementary proof of the Mazur-Ascoli theorem had been found earlier by M. H. Stone and others, see V. L. Klee, Jr., *Convex sets in linear spaces*, Duke Math. J., vol. 18 (1951), pages 443–66, 875–83 for references.)

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SECTION THREE

Bessel Expansions of the Confluent Hypergeometric Functions*

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Presented by R. D. JAMES, F.R.S.C.

Introduction

An initial or boundary value problem associated with a differential equation can be reformulated in terms of an integral equation in a variety of ways. When the problem cannot be solved exactly, such an integral equation is usually the starting point of any successive approximation procedure used to generate the solution. It is also natural to begin with such a reformulation when investigating the asymptotic behaviour of the solution, or, in the case of boundary value problems, when investigating the convergence of an eigenfunction expansion or the asymptotic behaviour of eigenvalues and eigenfunctions.

We shall first discuss briefly the reformulation of a general initial value problem in terms of an integral equation. We then adapt the method of successive substitutions to a special case and obtain the multiplication theorems for the Bessel functions. An extension of the idea leads to expansions of the confluent hypergeometric functions in series of Bessel functions.

We shall include references to the other applications mentioned above and to other expansions of the confluent hypergeometric functions which have been found by different methods.

The Integral Equation

Suppose that after dividing through by $p(z)$, the coefficients in the differential equation

$$\frac{d}{dz}\left(p(z)\frac{dy}{dz}\right) + q(z)\,y = r(z)\,y \tag{1}$$

*This investigation was carried out while two of the authors (C.A.S. and D.A.T.) held National Research Council Scholarships.

are analytic in the finite z-plane except perhaps for poles at the origin. Then, for $\epsilon \neq 0$,

$$(2)\quad y(z) = a\,v_1(z) + b\,v_2(z) + c\int_\epsilon^z \{v_1(x)\,v_2(z) - v_1(z)\,v_2(x)\}\, r(x)\,y(x)\,dx,$$

where v_1, v_2 are independent solutions of

$$(3)\qquad \frac{d}{dz}\left(p(z)\frac{dv}{dz}\right) + q(z)\,v = 0$$

and where a, b are arbitrary constants while

$$c = 1/p(x)\{v_1(x)\,v_2'(x) - v_2(x)\,v_1'(x)\}\,.$$

The transformation from (1) to (2) links the solutions of (1) with those of the associated equation (3).

The transformation can be generalized to apply to n-th order equations. For example, with suitable restrictions on the coefficients, the differential equation

$$(1')\qquad y^{(n)}(z) + \sum_{i=0}^{n-1} q_i(z)\,y^{(i)}(z) = \sum_{i=0}^{n-1} r_i(z)\,y^{(i)}(z)$$

can be transformed into

$$(2')\qquad y(z) = \sum_{i=1}^{n} a_i\,v_i(z) + \int_\epsilon^z K(z,x)\,y(x)\,dx,$$

where the v_i are independent solutions of

$$(3')\qquad v^{(n)}(z) + \sum_{i=0}^{n-1} q_i(z)\,v^{(i)}(z) = 0.$$

$K(z, x)$ can be found by the method of variation of parameters and it turns out to be

$$K(z,x) = \sum_{i=0}^{n-1} (-1)^i \left\{N(z,x)\,r_i(x)\right\}^{(i)}$$

where

$$\begin{vmatrix} v_1(x) & \ldots\ldots & v_n(x) \\ v_1'(x) & \ldots\ldots & v_n'(x) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ v_1^{(n-2)}(x) & \ldots & v_n^{(n-2)}(x) \\ v_1^{(n-1)}(x) & \ldots & v_n^{(n-1)}(x) \end{vmatrix} \; N(z,x) = \begin{vmatrix} v_1(x) & \ldots\ldots & v_n(x) \\ v_1'(x) & \ldots\ldots & v_n'(x) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ v_1^{(n-2)}(x) & \ldots & v_n^{(n-2)}(x) \\ v_1(z) & \ldots\ldots & v_n(z) \end{vmatrix}.$$

The transformation from (1) to (2) was first used by Liouville [10] in the special case when $p(z) = 1$, $q(z) = \rho^2$; the resulting integral equation was then used to study the asymptotic behaviour, for large ρ, of the eigenvalues and eigenfunctions determined by (1) after the imposition of certain boundary conditions. More general problems of the same nature have been considered by Birkhoff [2; 3].

Many papers have been devoted to a study of the asymptotic behaviour of solutions of differential equations. A majority of these papers have been based on a reformulation of the problem in terms of an integral equation and the pertinent theory has been developed chiefly during the past twenty years. Recent developments are given, for example, by Langer [14] and Cherry [7]. A transformation of an n-th order differential equation has been considered in detail by Miller [15].

It should be noted that the terms on the right side of (1) or (1′) need not be linear in y and its derivatives. If these terms are non-linear, but relatively small, the form (2) or (2′) is useful in finding successive approximations to the solution if the problem is an initial value problem. A trivial but interesting example of this fact arises in relativity theory where the path of light past the sun depends on the solution of [8]

$$\frac{d^2u}{d\phi^2} + u = 3mu^2,\ u(0) = \frac{1}{R},\ u'(0) = 0$$

where m is small. (Other methods are more direct when only the so-called limit cycle solutions of a non-linear problem are required.)

Multiplication Theorems for Bessel Functions

In the above mentioned studies of asymptotic expansions, the usual procedure has been to choose the integral equation (2) or (2′) so that the integral contained in it is of small order compared to the remaining terms, as some parameter approaches infinity. The remaining terms on the right side are then *asymptotic* to the desired solution $y(z)$. Apparently Ikeda [11] was the first to use the method of successive substitutions in (2) to obtain complete and *convergent* expansions of the solution in special cases. His results are not entirely correct; we shall therefore now treat his most important example in order to illustrate the procedure.

The differential equation for $J_{\pm\nu}(\alpha z)$ (ν not an integer) can be written in the form (1) as

$$\frac{d}{dz}\left(z\frac{dy}{dz}\right) + \left(z - \frac{\nu^2}{z}\right)y = z(1 - \alpha^2)\,y \tag{4}$$

In this case $v_1 = J_\nu(z)$ and $v_2 = J_{-\nu}(z)$, and (2) becomes

(5) $$y(z) = a\, J_\nu(z) + b\, J_\nu(z)$$
$$- \frac{\pi(1-\alpha^2)}{2 \sin \nu\pi} \int_\epsilon^z \{J_{-\nu}(x)\, J_{-\nu}(z) - J_\nu(z)\, J_{-\nu}(x)\}\, x\, y(x)\, dx.$$

Here and in later sections we need, for $i = 0, 1, 2, \ldots\ldots$, the results*

(6a) $$\frac{-\pi}{2 \sin \nu\pi} \int_\epsilon^z \{J_\nu(x)\, J_{-\nu}(z) - J_\nu(z)\, J_{-\nu}(x)\}\, x \{x^{i+1}\, J_{\nu+i}(x)\}\, dx$$
$$= \frac{z^{i+1}}{2(i+1)} J_{\nu+i+1}(z) + A_i J_\nu(z) + B_i J_{-\nu}(z),$$

where

(6b) $$A_i = \frac{-\pi\epsilon}{4(i+1)\sin \nu\pi}\left[J_{-\nu}(\epsilon) \frac{d}{d\epsilon} \{\epsilon^{i+1} J_{\nu+i+1}(\epsilon)\} - \{\epsilon^{i+1} J_{\nu+i+1}(\epsilon)\} \frac{d}{d\epsilon} J_{-\nu}(\epsilon) \right],$$

(6c) $$B_i = \frac{-\pi\epsilon}{4(i+1)\sin \nu\pi}\left[- J_\nu(\epsilon) \frac{d}{d\epsilon} \{\epsilon^{i+1} J_{\nu+i+1}(\epsilon)\} + \{\epsilon^{i+1} J_{\nu+i+1}(\epsilon)\} \frac{d}{d\epsilon} J_\nu(\epsilon) \right],$$

and

(7a) $$\frac{-\pi}{2 \sin \nu\pi} \int_\epsilon^z \{J_\nu(x)\, J_{-\nu}(z) - J_\nu(z)\, J_{-\nu}(x)\}\, x \{x^i\, J_{-\nu-i}(x)\}\, dx$$
$$= \frac{-z^{i+1}}{2(i+1)} J_{-\nu-i-1}(z) + C_i J_\nu(z) + D_i J_{-\nu}(z),$$

where

(7b) $$C_i = \frac{\pi\epsilon}{4(i+1)\sin \nu\pi}\left[J_{-\nu}(\epsilon) \frac{d}{d\epsilon} \{\epsilon^{i+1} J_{-\nu-i-1}(\epsilon)\} - \{\epsilon^{i+1} J_{-\nu-i-1}(\epsilon)\} \frac{d}{d\epsilon} J_{-\nu}(\epsilon) \right],$$

(7c) $$D_i = \frac{\pi\epsilon}{4(i+1)\sin \nu\pi}\left[- J_\nu(\epsilon) \frac{d}{d\epsilon} \{\epsilon^{i+1} J_{-\nu-i-1}(\epsilon)\} + \{\epsilon^{i+1} J_{-\nu-i-1}(\epsilon)\} \frac{d}{d\epsilon} J_\nu(\epsilon) \right].$$

*To avoid ambiguity in these results we will assume the z-plane to be cut, say along the negative real axis. The expansions obtained later will nevertheless hold without any restriction on arg z.

These results can be established as follows: it is a straight-forward matter to show that the left side of (6a) satisfies

$$\frac{d}{dz}\left(z\frac{du}{dz}\right) + \left(z - \frac{\nu^2}{z}\right) u = z^{i+1} J_{\nu+i}(z).$$

The first term on the right of (6a) is a particular integral of this equation and the other terms are the complementary function. A_i and B_i are found by putting $z = \epsilon$ in (6a) and in the equation obtained by differentiating (6a). The proof of (7) is analogous.

For each a and b, the method of successive substitutions applied to (5) will generate a series which converges to the solution of (5) for any finite z, $\epsilon \neq 0$. Assuming for the moment that we can rearrange the various contributions to the terms in this series, it is clear from (6) and (7) that we can write the solution in the form

$$(8) \qquad y(z) = \sum_{i=0}^{\infty} p_i z^i J_{\nu+i}(z) + \sum_{i=0}^{\infty} q_i z^i J_{-\nu-i}(z),$$

where p_i and q_i are constant coefficients.

Proceeding formally, we now substitute (8) into (5); using (6) and (7) and equating coefficients of like terms, we then find that (5) is satisfied if p_i and q_i satisfy the recurrence relations

$$(9) \qquad p_{i+1} = \frac{1-\alpha^2}{2(i+1)} p_i, \quad q_{i+1} = -\frac{1-\alpha^2}{2(i+1)} q_i, \quad i = 0, 1, 2, \ldots .$$

The values of p_0 and q_0 depend on a and b and so ultimately on the particular solution of (4) which is being considered.

For the solution to be $J_\nu(\alpha z)$, which is finite at the origin when $R(\nu) > 0$, we must take $q_0 = 0$ in (8). Equating coefficients of z^ν on both sides of (8), we obtain the value of p_0. From (9) the solution then formally reduces to

$$(10) \qquad J_\nu(\alpha z) = \alpha^\nu \sum_{i=0}^{\infty} \frac{(1-\alpha^2)^i}{i!} \left(\frac{z}{2}\right)^i J_{\nu+i}(z)$$

which we shall now show holds for all $R(\nu) \geqslant 0$.

Since [20, p. 44]

$$(11) \qquad |J_\nu(z)| \leqslant \left|\frac{(z/2)^\nu}{\Gamma(\nu+1)}\right| \exp \frac{|z|^2}{4|\nu_0+1|},$$

where $|\nu_0 + 1|$ is the smallest of the numbers $|\nu + 1|, |\nu + 2|, \ldots,$ the series (10) is dominated, when $R(\nu) > 0$, by .

$$\left|\frac{(\alpha z/2)^\nu}{\Gamma(\nu+1)}\right| \exp \frac{|z|^2}{4} \sum_{i=0}^{\infty} \frac{|1-\alpha^2|^i}{i!} \left|\frac{z}{2}\right|^{2i}$$

which obviously converges. The convergence is uniform with respect to z for z in any finite region, so our formal operation of integrating term by term with the aid of (6) is justified. It can be verified that the subsequent rearranging depends only on the absolute convergence of both (10) and the series obtained from (10) by differentiating. Requiring the coefficients of $J_\nu(z)$ and $J_{-\nu}(z)$ to vanish serves only to define a and b, and all the remaining coefficients vanish automatically. The convergence of (10) is also uniform with respect to ν, and hence we can let ν assume integral values. The formula (10) is therefore valid for all $R(\nu) \geqslant 0$.

For the solution of (5) to be $J_{-\nu}(\alpha z)$ ($R(\nu) > 0$ and ν not an integer), we find on equating coefficients of z^ν on both sides of (8) that $p_0 = 0$. Further, near $z = 0$,

$$J_{-\nu}(z) \approx \frac{1}{\Gamma(-\nu+1)}\left(\frac{z}{2}\right)^{-\nu} = \frac{\Gamma(\nu)\sin\nu\pi}{\pi}\left(\frac{z}{2}\right)^{-\nu}.$$

Using this result and (9) and putting $y = J_{-\nu}(\alpha z)$, we find, on equating coefficients of $(z/2)^{-\nu}$ in (8), that

$$\frac{\Gamma(\nu)\sin\nu\pi}{\pi}\alpha^{-\nu} = \frac{q_0}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i(1-\alpha^2)^i}{i!}\Gamma(\nu+i)\sin(\nu+i)\pi,$$

so that finally $q_0 = \alpha^\nu$ provided $|1-\alpha^2| < 1$.

Using (11), we find that the series for $J_{-\nu}(\alpha z)$ is dominated by

$$\left|\frac{(2\alpha)^\nu\Gamma(\nu)\sin\nu\pi}{\pi z^\nu}\right| \exp\frac{|z|^2}{4|\nu_0+1|}\sum_{i=0}^{\infty}\frac{|1-\alpha^2|^i|\Gamma(\nu+i)|}{i!\,|\Gamma(\nu)|}$$

which converges if $|1-\alpha^2| < 1$.

As before we therefore obtain, for $R(\nu) > 0$ and ν not an integer and $|1-\alpha^2| < 1$,

$$J_{-\nu}(\alpha z) = \alpha^\nu\sum_{i=0}^{\infty}\frac{(1-\alpha^2)^i}{i!}\left(\frac{-z}{2}\right)^i J_{-\nu-i}(z).$$

Moreover,

$$\begin{aligned}\frac{J_\nu(\alpha z)\cos\nu\pi - J_{-\nu}(\alpha z)}{\sin\nu\pi} &= \alpha^\nu\sum_{i=0}^{\infty}\frac{(1-\alpha^2)^i}{i!}\left(\frac{z}{2}\right)^i\frac{J_{\nu+i}(z)\cos\nu\pi - (-1)^iJ_{-\nu-i}(z)}{\sin\nu\pi}\\ &= \alpha^\nu\sum_{i=0}^{\infty}\frac{(1-\alpha^2)^i}{i!}\left(\frac{z}{2}\right)^i\frac{J_{\nu+i}(z)\cos(\nu+i)\pi - J_{-\nu-i}(z)}{\sin(\nu+i)\pi};\end{aligned}$$

that is,

$$(12) \qquad Y_\nu(\alpha z) = \alpha^\nu \sum_{i=0}^{\infty} \frac{(1-\alpha^2)^i}{i!} \left(\frac{z}{2}\right)^i Y_{\nu+i}(z).$$

Equation (12) therefore holds, at least when $R(\nu) > 0$ and ν is not an integer, so long as $|1 - \alpha^2| < 1$ and z is finite and different from zero. That (12) holds under the same restrictions on α and z even when ν is an integer n, follows from the fact that

$$Y_{n+i}(z) = O((n+i)\,!)$$

(so that the series converges when $\nu = n$) and from the continuity of Y_ν as a function of ν.

Equations (10) and (12) are the so-called multiplication theorems for Bessel functions [20, p. 142].

Expansions of the Confluent Hypergeometric Functions

We shall begin with the differential equation for the confluent hypergeometric functions $M_{k,\pm m}(y)$ given by Whittaker [21, p. 337]:

$$(13) \qquad \frac{d^2W}{dy^2} + \left\{-\frac{1}{4} + \frac{k}{y} + \frac{\frac{1}{4} - m^2}{y^2}\right\} W = 0,$$

where $2m$ is assumed to be not an integer. The change of variables and parameters

$$(14) \qquad 4ky = z^2, \qquad W(y) = zF(z), \qquad 2m = \nu$$

transforms (13) into the form of (1):

$$\frac{d}{dz}\left(z\frac{dF}{dz}\right) + \left(z - \frac{\nu^2}{z}\right) F = \frac{z^3}{16k^2} F$$

where ν is not an integer, so that $v_1 = J_\nu(z)$, $v_2 = J_{-\nu}(z)$, and the corresponding integral equation for F becomes

$$(15) \quad F(z) = a\, J_\nu(z) + b\, J_{-\nu}(z)$$

$$- \frac{\pi}{32\, k^2 \sin \nu\pi} \int_0^z \{J_\nu(x)\, J_{-\nu}(z) - J_\nu(z)\, J_{-\nu}(x)\}\, x^3 F(x)\, dx.$$

Rearranging the terms in the series generated by the successive substitution procedure can again lead to a solution of the form (8). On putting (8) into the integral equation (15) we find, however, that before each integration can be performed with the aid of (6) and (7), it is necessary to use a recurrence relation on each term of the series.

Precisely, we rewrite (8) as

$$F(z) = \sum_{i=0}^{\infty} p_i z^i \left[2\frac{\nu+i+1}{z} J_{\nu+i+1}(z) - J_{\nu+i+2}(z) \right]$$
$$+ \sum_{i=0}^{\infty} q_i z^i \left[-2\frac{\nu+i+1}{z} J_{-\nu-i-1}(z) - J_{-\nu-i-2}(z) \right]$$

before substitution into the integral of (15). The relations (6) and (7) can then be applied. Proceeding formally, and putting $p_i = p_0\bar{p}_i$, $q_i = q_0\bar{q}_i$, $\bar{p}_0 = \bar{q}_0 = 1$ we obtain, for $i = 0, 1, 2, \ldots\ldots$, the relations

$$\text{(16)} \qquad \begin{cases} \bar{p}_1 = 0, \quad \bar{p}_2 = \dfrac{1}{32k^2}(\nu+1), \\ \bar{p}_{i+3} = \dfrac{1}{32k^2}\left\{2\dfrac{\nu+i+2}{i+3}\bar{p}_{i+1} - \dfrac{\bar{p}_i}{i+3}\right\} \end{cases}$$

and

$$\text{(17)} \qquad \begin{cases} \bar{q}_1 = 0, \quad \bar{q}_2 = \dfrac{1}{32k^2}(\nu+1), \\ \bar{q}_{i+3} = \dfrac{1}{32k^2}\left\{2\dfrac{\nu+i+2}{i+3}\bar{q}_{i+1} + \dfrac{\bar{q}_i}{i+3}\right\}. \end{cases}$$

Arguing as we did to establish the result (10) for $J_\nu(\alpha z)$, and returning to the original variables through (14), we obtain

$$\text{(18)} \qquad M_{k,m}(y) = \tfrac{1}{2} k^{-m-\frac{1}{2}}\Gamma(2m+1)\sum_{i=0}^{\infty} \bar{p}_i (2\sqrt{ky})^{i+1} J_{2m+i}(2\sqrt{ky})$$

which is valid for all k, finite y, and for $R(m) \geqslant 0$; the $\bar{p}_i$ are given by (16). Using Langer's method, Taylor [18, Eq. (33)] has shown that the first term of this series is asymptotic to $M_{k,m}(y)$ as $|k| \to \infty$.

The corresponding result for $M_{k,-m}(y)$ with $R(m) > 0$ turns out to be an asymptotic expansion valid for large $|k|$. It is

$$\text{(19)} \qquad M_{k,-m}(y) \sim q_0 \sum_{i=0}^{\infty} \bar{q}_i (2\sqrt{ky})^{i+1} J_{-2m-i}(2\sqrt{ky})$$

for y finite and different from zero; the $\bar{q}_i$ are given by (17).

We shall first indicate how to prove that (19) is an asymptotic expansion valid for large $|k|$, and then find q_0 (see equation (21)). Substituting the expression

$$\text{(20)} \quad F(z) = p_0 \sum_{i=0}^{n} \bar{p}_i z^i J_{\nu+i}(z) + q_0 \sum_{i=0}^{n} \bar{q}_i z^i J_{-\nu-i}(z) + f(z)$$

into equation (15) and carrying out the integrations over the partial sums, we obtain an integral equation for $f(z)$. Using the recurrence relations (16) and (17) and choosing a, b or p_0, q_0 so that the co-

efficients of J_ν and $J_{-\nu}$ cancel out, we finally reduce the equation for $f(z)$ to the form

$$f(z) = \frac{p_0}{16k^2}[\bar{p}_{n-2}(\ldots) + \bar{p}_{n-1}(\ldots) + \bar{p}_n(\ldots)]$$

$$+ \frac{q_0}{16k^2}[\bar{q}_{n-2}(\ldots)\ldots] - \frac{\pi}{32k^2 \sin \nu\pi}\int_c^z \{J_\nu(x)J_{-\nu}(z) - J_\nu(z)J_{-\nu}(x)\} x^3 f(x)\, dx.$$

Now, if M denotes the maximum of $|f(x)|$ along the path of integration, then

$$M \leqslant (|p_0| A + |q_0| B + MC)/|k|^2$$

where A, B, C are upper bounds for corresponding expressions in the preceding equation. Therefore

$$M \leqslant \frac{|p_0|A + |q_0| B}{|k|^2(1 - C/|k|^2)}$$

if $|k|$ is chosen large enough that the denominator is positive. From (16) and (17), both $\bar{p}_i$, $\bar{q}_i$ are $O(|k|^{-2i/3})$, so that

$$M = p_0\, O(|k|^{-2(n+1)/3}) + q_0\, O(|k|^{-2(n+1)/3});$$

moreover, the last term in each of the partial sums on the right of (20) is $O(|k|^{-2n/3})$. The series obtained from (20) is therefore asymptotic to the solution of (15) for large $|k|$.

Equating coefficients of z^ν and $z^{-\nu}$ in the $F(z)$ corresponding to $M_{k,-m}(y)$ [21, p. 338] and in the right side of (20), we obtain

$$p_0 = 0$$

$$(4k)^{(\nu-1)/2} \sim q_0(\sin \nu\pi/\pi)\sum_{i=0}^{\infty}(-1)^i \bar{q}_i \Gamma(\nu + i).$$

Hence, returning to the original variables, we obtain an asymptotic expansion for the coefficient q_0 in (19):

$$(21) \qquad 1/q_0 \sim (4k)^{-m+\frac{1}{2}}(\sin 2m\pi/\pi)\sum_{i=0}^{\infty}(-1)^i \bar{q}_i \Gamma(2m + i).$$

Appropriate changes of variables in (18) and (21) lead to expansions of, for example, the Laguerre and the Hermite functions.

Concluding Remarks

We have shown that the usual method of studying the asymptotic solutions of differential equations can, with some modification, be used to obtain complete expansions of special functions in certain

important cases. In a future paper we propose to adapt the method to the solution of certain boundary value problems—the so-called bounded quantum mechanical problems.

Kuhn [13] presents considerable analytic and numerical detail obtained from expansions of the confluent hypergeometric functions in series of Bessel functions. His expansions can be obtained from the integral equation (15) by successive substitutions without any subsequent rearrangement of terms except for collecting the J_ν and $J_{-\nu}$ terms.

Using a variety of methods, other expansions of the confluent hypergeometric functions have been found by Abramowitz [1], Buchholz [4], Henrici [9], Karlin [12], Pignedoli [17], and Tricomi [19].

Asymptotic expansions of the confluent hypergeometric functions are given or referred to by Chang, Chu and O'Brien [6]. Such expansions have also been considered by Moecklin [16] and Buchholz [5].

We wish to express our appreciation of many interesting and helpful discussions with Dr. G. E. Latta.

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TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA
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SECTION THREE

The Ergodic Theorem for Banach Spaces with Convex-Compactness

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1. Introduction

LET T be a bounded, linear operator on a real or complex Banach space B and let

$$T_n = \frac{1}{n+1}\,(1 + T + T^2 + \ldots + T^n).$$

T is said to be ergodic on B in the sense of J. von Neumann if for each f in B, $T_n f$ converges to a limit when n becomes infinite.

In order that T be ergodic on B, the following conditions are necessary:

(1.1) $(1/n)\; T^n f \to 0$ when $n \to \infty$, for every f in B,

(1.2) $|\, T_n \,| \leqslant K$ for all n, for some $K < \infty$.

Indeed, (1.1) follows from the relation

$$(1.3) \qquad (1/n)\; T^n = \frac{n+1}{n}\, T_n - T_{n-1}$$

and (1.2) from the theorem that a convergent sequence of bounded, linear operators must be uniformly bounded [1, p. 80, théorème 5].

Clearly, if T is ergodic on B and $T_n f$ converges to f^*, say, then (1.1) and the identity

$$T_n(Tf) = T(T_n f) = T_n f + \frac{1}{n+1}\,(T^{n+1}f - f)$$

together with the continuity of T, show that $Tf^* = f^*$.

Because of (1.3), (1.2) implies that $|\,(1/n)T^n\,| \leqslant 3K$ for all n and hence that $(1/n)T^n f$ will converge to 0 for all f if $(1/n)T^n f$ converges to zero for a set of f which span B, that is, whose linear combinations are dense in B.

Both (1.1) and (1.2) are implied by the condition:

(1.4) $|\, T^n \,| \leqslant K$ for all n, for some $K < \infty$.

But if T is ergodic on B and (1.4) holds, then the identity

$$T_n(T^r f) - f^* = T^r(T_n f - f^*)$$

shows that $T_n(T^r f)$ converges to f^*, irrespective of how the non-negative integer r varies, when n becomes infinite.

In this note we discuss a method of F. Riesz [6; 7] to show that the conditions (1.1) and (1.2) are sufficient to imply that T is ergodic on B if B is *convex-compact*. By definition, we shall say that a subset A of B (possibly all of B) is convex-compact if every decreasing sequence of non-empty, closed, bounded, *convex* subsets of A has a non-empty intersection.

2. The Riesz Method of Averages

Suppose that B is an arbitrary Banach space and that T satisfies (1.1) and (1.2). Let $S(f)$ denote the set of all "averages" of the form

$$g = c_0 f + c_1 Tf + \ldots + c_r T^r f$$

where r is an arbitrary non-negative integer and the c_i are non-negative scalars with sum equal to 1. Riesz observed that for such g and for $n > r$,

$$T_n f - T_n g = \frac{1}{n+1} \sum_{j=0}^{r-1} (c_{j+1} + \ldots + c_r)(T^j f - T^{n+1+j} f).$$

Hence (1.1) implies

$$\overline{\lim_{n\to\infty}} \, |T_n f - T_n g| = 0,$$

and therefore

$$\begin{aligned} \overline{\lim_{n,m\to\infty}} \, |T_n f - T_m f| &\leqslant \overline{\lim_{n\to\infty}} \, |T_n f - T_n g| + \overline{\lim_{n\to\infty}} \, |T_n g - g| \\ &\quad + \overline{\lim_{m\to\infty}} \, |g - T_m g| + \overline{\lim_{m\to\infty}} \, |T_m g - T_m f| \\ &= 2 \, \overline{\lim_{n\to\infty}} \, |T_n g - g|. \end{aligned}$$

Thus, following Riesz, T will be ergodic on B if there are elements g_i in $S(f)$ such that

(2.1) $$\overline{\lim_{n\to\infty}} \, |T_n g_i - g_i| \to 0 \qquad \text{when } i \to \infty.$$

Now there will be such g_i in $S(f)$ if there is an element h in $\overline{S(f)}$, the closure of $S(f)$, with $Th = h$. Indeed, there will then be g_i in $S(f)$ such that $| g_i - h | \to 0$ when $i \to \infty$; since

$$T_n g_i - g_i = T_n(g_i - h) + (T_n h - h) + (h - g_i),$$

the identity $T_n h = h$ together with (1.2) shows that (2.1) will hold.

Thus T will be ergodic on B if for every f, the mapping of $\overline{S(f)}$ into itself by T has a fix-point.

Now let $S_n(f)$ be the set of elements u in $\overline{S(f)}$ satisfying both $|u| \leqslant K|f|$ and $|Tu - u| = |(T-1)u| \leqslant 1/n$. Then every $S_n(f)$ is non-empty; for $|T_m f| \leqslant K|f|$ for all m by (1.2), and the identity

$$\left| T(T_m f) - T_m f \right| = \frac{1}{m+1}\left| T^{m+1}f - f \right| \leqslant \frac{1}{m+1}\left| T^{m+1}f \right| + \frac{1}{m+1}\left| f \right|$$

together with (1.1) show that, for fixed n, $T_m f$ is contained in $S_n(f)$ for all sufficiently large m. It is easily verified that for fixed f, the $S_n(f)$ are a non-increasing sequence of bounded, closed, convex sets. If B is convex-compact then some element h will be in all $S_n(f)$; clearly, this h will be in $\overline{S(f)}$ and for every n, $|Th - h| \leqslant 1/n$ so that $Th = h$.

This proves: *if B is convex-compact then a bounded, linear operator T is ergodic on B if and only if T satisfies* (1.1) *and* (1.2).

Since metrically closed convex subsets of B are necessarily weakly closed (this is a theorem of Ascoli-Mazur, see Halperin, these Transactions, vol. 47, pp. 1–6) weak convex-compactness is equivalent to the strong convex-compactness as defined above. Hence our ergodic theorem includes the case of locally weakly compact Banach spaces [8] and the equivalent case of reflexive Banach spaces [5], equivalent by the theorem of Eberlein. These writers assume (1.4) but their arguments are actually valid assuming only (1.1) and (1.2). The reflexive spaces are now known to include the uniformly convex Banach spaces of Clarkson [2], and in particular, the L^p spaces, $p > 1$. We refer to [8] for references to other writers on this subject but particular reference should be made to the proof of an ergodic theorem by Riesz [7].

3. The Uniformly Convex Banach Space

We note that it is particularly easy to show that every uniformly convex B is convex-compact. Indeed, if S_n are a decreasing sequence of non-empty, bounded, closed, convex sets, let $\mu_n =$ infimum $|g|$ for all g in S_n, and let g_n be selected from S_n with $\mu_n \leqslant |g_n| \leqslant \mu_n + 1/n$. Then the μ_n are non-decreasing and bounded above since the S_n are bounded. It follows that $|g_n|$ converges to a limit as n becomes infinite and the convexity of the S_n together with the uniform convexity of B implies that the g_n converge to a limit h. This h is necessarily in all S_n, assumed closed.

4. Additional Comments

Call an element f in the Banach space T-ergodic if $T_n f$ converges when n becomes infinite. Then (1.2) implies that the T-ergodic

elements form a closed linear subspace of B. Our argument actually shows that f is T-ergodic if $\overline{S(f)}$ is convex-compact, and that T is ergodic on B if $\overline{S(f)}$ is convex-compact for a set of f which span B, assuming (1.1) and (1.2).

Finally, consider the special case that B is L^1: the space of $f(P)$ summable on a set X of points P with a non-negative, countably additive measure $|e|$, and that T is a bounded, linear operator of the type

$$Tf(P) = f(\phi(P))$$

for some mapping $\phi(P)$ of X into itself. It is easy to see that L^1 is not convex-compact, except when X consists essentially of a finite number of points. However, if X has finite measure and T is of this special type then for every measurable subset e the function 1_e, which is defined to have value 1 if P is in e, and value 0 otherwise, has $S(1_e)$ convex-compact, in fact every sequence of elements g_n from $S(1_e)$ has a weakly convergent subsequence. This follows from the fact that the g_n are uniformly bounded ($|g_n(P)| \leqslant 1$ for all P) and X is of finite measure. Thus (1.1) and (1.2) imply that such a T is ergodic on such an L^1(see [4]); but (1.1) may be omitted since it is automatically valid for all 1_e when X is of finite measure, and the 1_e span L^1; also, (1.2) can be deduced if $|T_n 1_e| \leqslant K|e|$ is known to be valid for all e, as pointed out by Dunford and Miller [3]. Thus the necessary and sufficient condition that a T of this type be ergodic on such an L^1 is that for some $K < \infty$ and for all n and all e, (see [3]),

$$\frac{1}{n+1}(|e| + |\phi^{-1}(e)| + \ldots + |\phi^{-n}(e)|) \leqslant K|e|.$$

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SECTION THREE

A Problem in Combinatorial Analysis

N. S. MENDELSOHN

Presented by R. L. JEFFERY, F.R.S.C.

1. Introduction

PROBLEMS in combinatorial analysis are usually considered solved when one can find either a difference equation or a recurrence relationship which enables one to compute $f(n)$ step by step. If it makes computation easier, an explicit formula for $f(n)$ is a more desirable type of solution. In common with other branches of number theory, certain published "solutions" of combinatorial problems merely replace the original problem by another one which may in fact be more difficult to solve than the original problem. The author of this paper has discovered in [1] and [2] a symbolic method of setting up and solving the recurrence relations for a wide class of combinatorial problems. The problem studied in this paper can be handled by the author's general method, but the computations are very laborious. However, the formula, once obtained, can be verified much more easily and in this paper we will present only this easy verification. Numerical examples will illustrate the computational efficiency of our formula.

2. The Problem

We will consider the following problem. Suppose we have a set of $a + b + c + \ldots$ objects of which a are alike of one kind, b alike of a second kind, etc. Let $C_r[a, b, c, \ldots]$ denote the number ofcombinations taken r at a time of these objects. We will write $C_r[a^3, b^2, c, \ldots]$ as an abbreviation for $C_r[a, a, a, b, b, c, \ldots]$. An explicit formula for

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}]$$

will be obtained. For the corresponding permutation problem, i.e., the problem of finding the number of permutations r at a time of the given class of objects, a recurrence formula will be obtained. Corresponding to the notation $C_r[a, b, c, \ldots]$ we will use $P_r[a, b, c, \ldots]$ in the case of the permutation problem.

3. The Fundamental Recurrence

We first show that

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}]$$

satisfies the following recurrence relationship:

$$(1) \qquad C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}] = \sum_{u=0}^{\min(r, a_t)} C_{r-u}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t - 1}]$$

To prove this formula we isolate one set S of a_t identical objects and observe how these objects enter into the combinations. The combinations which are counted by the expression

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}]$$

can be divided into the following mutually exclusive classes:

(0) Those which contain no object of the set S. There are

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t - 1}]$$

of these.

(1) Those which contain one object of the set S. There are

$$C_{r-1}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t - 1}]$$

of these.

. . . .

(u) Those which contain u objects of the set S. There are

$$C_{r-u}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t - 1}]$$

of these.

. . . .

On adding the expressions obtained in (0), (1), . . . , (u), . . . we obtain formula (1).

Formula (1) is quite efficient for computing purposes but an explicit expression will be given in the next section which is actually much better.

4. The Explicit Formula

An explicit formula for

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}]$$

can now be obtained. However, we will write down the formula for the case $C_r[a^i, b^j]$ only. A simple examination of this formula will

show how it may be extended to the general case. The formula in question is:

$$\text{(II)} \quad C_r[a^i, b^j] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m} \binom{i+j+r-1-(a+1)k-(b+1)m}{r-(a+1)k-(b+1)m}$$

where the summation is taken over all integral values of the indices k, m satisfying the inequalities: $0 \leqslant k \leqslant i, 0 \leqslant m \leqslant j, 0 \leqslant (a+1)k + (b+1)m \leqslant r$. In practice, it will be seen that these inequalities confine k and m to a very small range. In the next section we will show that formula (II) is quite easy to handle numerically. To prove formula (II), we first note that if $j = 0$, (II) is replaced by the formula

$$\text{(III)} \qquad C_r[a^i] = \sum_k (-1)^k \binom{i}{k}\binom{i+r-1-(a+1)k}{r-(a+1)k}$$

summed over all values of k in the interval

$$0 \leqslant k \leqslant \min\left(i, \left[\frac{r}{a+1}\right]\right)$$

where $[X]$ stands for the greatest integer less than or equal to X. When $i = 1$, (III) becomes $C_r[a] = 1$, if $r \leqslant a$, and $C_r[a] = 0$ for $r > a$. This result is obviously true, so that (III) is true if $i = 1$. The recurrence (I) for the case $t = 1$, $k_1 = i$ is

$$\text{(IV)} \qquad C_r[a^i] = \sum_{u=0}^{\min(r,a)} C_{r-u}[a^{i-1}].$$

Assuming (III) to be true when i is replaced by $i-1$, and using (IV) one obtains that (III) is true for i if it is true for $i-1$. Hence (III) is true for all integral i, by induction. Now $C_r[a^i] = C_r[a_i, b^0]$ so that (II) is true when $j = 0$. By a similar application of (I) to the case $t = 2$ we can establish (II) by induction, to be true for all j. This completes the proof.

One can obtain simpler formulae in case any of the a_i are greater than or equal to r. In this case the number of combinations is actually independent of the exact value of a_i. We will replace a_i by the symbol ∞ in case $a_i \geqslant r$ so that, for example, $C_5[2^3, 3^2, 5^4, 6^8, 7^9]$ will be written as $C_5[2^3, 3^2, \infty^{21}]$. The formula for $C_r[a^i, b^j, \infty^p]$ can be verified to be

$$\text{(V)} \quad C_r[a^i, b^j, \infty^p] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m} \binom{i+j+r+p-1-(a+1)k-(b+1)m}{r-(a+1)k-(b+1)m}$$

where the summation indices k, m range over all integers satisfying: $0 \leqslant k \leqslant i$, $0 \leqslant m \leqslant j$, $0 \leqslant (a+1)k + (b+1)m \leqslant r$.

In the next section we will illustrate the efficacy of these formulae by numerical examples.

5. Some Illustrative Examples

As a first example we will consider the number of essentially different 11-card canasta hands which can be formed from a 104-card pack. (We do not count the red threes as part of the pack since these are always replaced in a hand.) The pack consists of four black threes, four jokers, eight aces, eight twos, eight fours, ... , eight kings. Cards of a given denomination are considered equivalent regardless of suit so that the number of ways of choosing eleven cards from such a pack is $C_{11}[8^{12}, 4^2]$. Substituting in (II) we obtain

$$C_{11}[8^{12}, 4^2] = \sum_k \sum_m (-1)^{k+m} \binom{12}{k}\binom{2}{m}\binom{24-9k-5m}{11-9k-5m}$$

summed over all pairs of integers (k, m) subject to the inequalities $0 \leqslant k \leqslant 12$, $0 \leqslant m \leqslant 2$, $0 \leqslant 9k + 5m \leqslant 11$. The pairs (k, m) satisfying these inequalities are: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$. Hence,

$$C_{11}[8^{12}, 4^2] = \binom{24}{11} - 2\binom{19}{6} + 14 - 12\binom{15}{2} = 2{,}440{,}634.$$

It will be noted that our solution requires the computation of four terms only. A solution based on the partitions of eleven would require the computation of 52 terms each of which is a product of binomial coefficients. A solution by J. P. Ballantyne [3] would require the construction of a table containing 12 rows and 16 columns, i.e., one containing 192 entries.

As a second example, we consider the problem of finding the number of essentially different bridge hands. By this we mean the number of different bridge hands which can be formed when the cards in each suit from 2 to 9 inclusive are ranked equally. The required number is $C_{13}[8^4, 1^{20}]$. By (II)

$$C_{13}[8^4, 1^{20}] = \sum_k \sum_m \binom{4}{k}\binom{20}{m}\binom{36-9k-2m}{13-9k-2m}(-1)^{k+m}$$

summed over all pairs (k, m) such that $0 \leqslant k \leqslant 4$, $0 \leqslant m \leqslant 20$, $0 \leqslant 9k + 2m \leqslant 13$. Hence, the admissible (k, m) are: $(0, 0)$, $(0,1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, $(0, 5)$, $(0, 6)$, $(1, 0)$, $(1, 1)$, $(1, 2)$. This yields

$$C_{13}[8^4, 1^{20}] = \binom{36}{13} - 20\binom{34}{11} + 190\binom{32}{9} - 1140\binom{30}{7} + 4845\binom{28}{5}$$
$$- 15504\binom{26}{3} + 38760\binom{24}{1} - 4\binom{27}{4} + 80\binom{25}{2} - 760$$
$$= 34{,}033{,}880.$$

Here again, the saving in computation is very considerable when compared with the usual counting methods. This result is in agreement with that given by Ballantyne in [3] and affords a comparison between the two methods.

6. The Corresponding Permutation Problem

We will denote by $P_r[a, b, c, \ldots]$ the number of permutations taken r at a time of $a + b + c + \ldots$ objects of which a are alike of one kind, b are alike of another kind, etc. We will use the notation

$$P_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}]$$
$$\text{and } P_r[a^i, \infty^j]$$

to represent the permutation analogues of

$$C_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}] \text{ and } C_r[a^i, \infty^j].$$

We will confine ourselves mainly to the problem of finding the permutation analogue of (I) namely:

$$\text{(VI)} \quad P_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}] = \sum_{u=0}^{\min(r, a_t)} \binom{r}{u} P_{r-u}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t - 1}]$$

Before proving (VI) we note that the formulae $P_r[\infty^i] = i^r$ and

$$P_r[1^s] = \frac{s!}{(s-r)!}$$

are easily obtained from (VI) by induction. These formulae are, in fact, classical results.

The proof of (VI) is analogous to the proof of (I). We isolate one set S of a_t identical objects and then divide the totality of permutations into the following exclusive classes: For $\nu = 0, 1, 2, 3, \ldots, \min(r, a_t)$, we form the class $P_{(\nu)}$ of all permutations containing exactly ν elements of the set S. Then, obviously,

$$P_r[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t}] = \sum_{\nu=0}^{\min(r, a_t)} P_{(\nu)}.$$

To compute $P_{(\nu)}$ we note that each permutation in the set $P_{(\nu)}$ can be obtained by forming a permutation from $r - \nu$ objects chosen from

the totality of objects except those in S and then introducing into this permutation ν objects taken from S. The number of ways of introducing these ν objects into the permutation is

$$\binom{r}{\nu},$$

and the number of permutations into which these objects must be introduced is

$$P_{r-\nu}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t-1}].$$

Hence,

$$P_{(\nu)} = \binom{r}{\nu} P_{r-\nu}[a_1^{k_1}, a_2^{k_2}, \ldots, a_t^{k_t-1}].$$

This completes the proof of formula (VI).

As yet, the author has been unable to transform the recurrence (VI) into an explicit formula. In spite of this, formula (VI) gives rise to a rapid computational procedure, and in special cases does lead to explicit formulae.

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TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA
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SECTION THREE

Note on a Combinatorial Formula of Mendelsohn

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Presented by R. L. JEFFERY, F.R.S.C.

THE object of this note is to give a short, direct proof of result (II) of the preceding paper. The result in question is

$$\text{(II)}\quad C_r\,[a^i, b^j] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m}\binom{i+j+r-1-(a+1)\,k-(b+1)\,m}{r-(a+1)\,k-(b+1)\,m}$$

where the summation ranges over all values of k and m for which the binomial coefficients involved are positive.

Using the set up of Ballantyne's paper [3], we have $C_r[a^i, b^j]$ is the coefficient of x^r in

$$\begin{aligned} f(x) &= (1+x+x^2+\ldots+x^a)^i(1+x+x^2\ldots+x^b)^j \\ &= (1-x^{a+1})^i\,(1-x^{b+1})^j\,(1-x)^{-(i+j)} \end{aligned}$$

Now

$$(1-x)^{-(i+j)} = \sum_{t\geqslant 0}\binom{i+j-1+t}{t}x^t$$

Hence

$$C_r\,[a^i, b^j] = \sum_k \sum_m (-1)^{k+m}\binom{i}{k}\binom{j}{m}\binom{i+j-1+t}{t}$$

where t is determined by $r = (a+1)k + (b+1)m + t$. This completes the proof of (II).

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Some Remarks on Laplace's Method

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Presented by M. WYMAN, F.R.S.C.

IN his book, *The Laplace Transform*, D. V. Widder finds asymptotic formulae for integrals of the form

$$\int_a^b \exp\,(kh(x))\ \phi(x)\,dx$$

as $k \to \infty$, under the conditions that $h(x)$ has a simple flat maximum at a point $x = c$ between a and b, i.e. $h'(c) = 0$, $h''(c) < 0$, and $\phi(x)$ has, at worst, a simple zero at $x = c$ and does not change sign there, together with certain other conditions.

These results can easily be generalized to the case in which $h(x)$ has a higher order maximum at $x = c$, and $\phi(x)$ has a zero of arbitrary order there, though $\phi(x)$ is still not allowed to change sign at $x = c$. The generalization is contained in the following two theorems.

The first theorem deals with the case in which the maximum of $h(x)$ occurs at one end point of the interval, and the second theorem with the case in which the maximum of $h(x)$ occurs at an interior point of the interval.

THEOREM 1. *If*

(1) $a < a + \eta < b$,

(2) $h(x) \in C^{2m}(a \leqslant x \leqslant a+\eta)$, $h^{(n)}(a) = 0$, $(n = 1, 2, \ldots, 2m-1)$, $h^{(2m)}(a) < 0$, $h(x)$ *is non-increasing in* $a \leqslant x \leqslant b$,

(3) $(x-a)^{\nu}\ \phi(x) \in L(a,b)$, $\phi(a+)$ *exists and* $\phi(a+) \neq 0$,

then

$$\int_a^b \exp\,(kh(x))(x-a)^{\nu}\ \phi(x)\,dx$$

$$\sim (-(2m)!/kh^{(2m)}(a))^{(\nu+1)/2m}(\phi(a+\ )/2m)\exp\,(kh(a))\Gamma((\nu+1)/2m)$$

as $k \to \infty$.

Proof. We may assume $\phi(a+) > 0$. Then for every ϵ such that $0 < \epsilon < \phi(a+)$ and $0 < \epsilon < -h^{(2m)}(a)$, δ exists such that

(i) $0 < A_1 = \phi(a+) - \epsilon < \phi(x) < A_2 = \phi(a+) + \epsilon$,

(ii) $B_1 = h^{(2m)}(a) - \epsilon < h(x) < B_2 = h^{(2m)}(a) + \epsilon < 0$, for all x, $a \leqslant x \leqslant a + \delta$.

Let

$$I_k = \int_a^b \exp\{k(h(x) - h(a))\}(x-a)^\nu \phi(x)\,dx$$
$$= \int_a^{a+\delta} + \int_{a+\delta}^b = I_k' + I_k''.$$

Then

$$|I_k''| \leqslant \exp\{k(h(a+\delta) - h(a))\} \int_a^b |\phi(x)|(x-a)^\nu\,dx.$$

Since $h(a+\delta) - h(a) < 0$, it is clear that $I_k'' = O(\alpha^k)$, $0 < \alpha < 1$.

Using (i) and Taylor's theorem with remainder, we have

$$A_1 \int_a^{a+\delta} \exp\{kh^{(2m)}(\xi)(x-a)^{2m}/(2m)!\}(x-a)^\nu\,dx < I_k'$$
$$< A_2 \int_a^{a+\delta} \exp\{kh^{(2m)}(\xi)(x-a)^{2m}/(2m)!\}(x-a)^\nu\,dx,$$

where $a < \xi < a + \delta$. From (ii) we only strengthen these inequalities by replacing $h^{(2m)}(\xi)$ by $h^{(2m)}(a) - \epsilon$ on the right and by $h^{(2m)}(a) + \epsilon$ on the left. Thus

$$A_1 \int_a^{a+\delta} \exp\{kB_1(x-a)^{2m}/(2m)!\}(x-a)^\nu\,dx < I_k'$$
$$< A_2 \int_a^{a+\delta} \exp\{kB_2(x-a)^{2m}/(2m)!\}(x-a)^\nu\,dx.$$

In the left hand side of these inequalities let

$$-u = kB_1(x-a)^{2m}/(2m)!,$$

and in the right hand side, let

$$-u = kB_2(x-a)^{2m}/(2m)!.$$

Thus the inequalities become

$$(-(2m)!/kB_1)^{(\nu+1)/2m}(A_1/2m) \int_0^{kC_1} e^{-u} u^{((\nu+1)/2m)-1}\,du < I_k'$$
$$< (-(2m)!/kB_2)^{(\nu+1)/2m}(A_2/2m) \int_0^{kC_2} e^{-u} u^{((\nu+1)/2m)-1}\,du$$

where $C_i = -B_i\delta^{2m}/(2m)!$ $(i = 1, 2)$.

Thus,

$$(-(2m)!/B_1)^{(\nu+1)/2m}(A_1/2m)\,\Gamma((\nu+1)/2m) \leqslant \underline{\overline{\lim}}\, I_k'$$
$$\leqslant (-(2m)!/B_2)^{(\nu+1)/2m}(A_2/2m)\,\Gamma((\nu+1)/2m).$$

Hence, since ϵ is arbitrary,

$$\lim_{k\to\infty} k^{(\nu+1)/2m} I_k = \lim_{k\to\infty} k^{(\nu+1)/2m} I_k{}'$$
$$= (-(2m)!/h^{(2m)}(a))^{(\nu+1)/2m}(\phi(a+)/2m)\ \Gamma((\nu+1)/2m).$$

Corollary 1. *If*

(1) $a < b - \eta < b$,

(2) $h(x) \in C^{2m}(b - \eta \leqslant x \leqslant b), h^{(n)}(b) = 0, (n = 1, 2, \ldots, 2\mathrm{m}-1)$, $h^{(2m)}(b) < 0$, $h(x)$ *is non-decreasing in* $a \leqslant x \leqslant b$,

(3) $(b - x)^{\nu}\phi(x) \in L(a, b)$, $\phi(b-)$ *exists and* $\phi(b-) \neq 0$,

then,

$$\int_a^b \exp(kh(x))(b - x)^{\nu}\phi(x)\,dx$$
$$\sim(-(2m)!/kh^{(2m)}(b))^{(\nu+1)/(2m)}(\phi(b-)/2m)\exp(kh(b))\ \Gamma((\nu+1)/2m)$$

as $k \to \infty$.

Corollary 2. *If* (1), (2), *and* (3) *of Theorem 1 hold, with the exception that* $\phi(a+) = 0$, *then*

$$\int_a^b \exp(kh(x))(x - a)^{\nu}\phi(x)\,dx = o(k^{-(\nu+1)/2m}\exp(kh(a))).$$

Theorem 2. *If*

(1) $a < b - \eta < b < b + \eta < c$,

(2) $h(x) \in C^{2m}(b - \eta \leqslant x \leqslant b + \eta)$, $h^{(n)}(b) = 0$, $(n = 1, 2, \ldots, 2m - 1)$, $h^{(2m)}(b) < 0$, $h(x)$ *is non-decreasing in* $a \leqslant x \leqslant b$, *and non-increasing in* $b \leqslant x \leqslant c$,

(3) $|x - b|^{\nu}\phi(x) \in L(a, c)$, $\phi(b+)$, $\phi(b-)$ *exist and are not zero.*

then

$$\int_a^c \exp(kh(x))|x - b|^{\nu}\phi(x)\,dx$$
$$\sim(-(2m)!/kh^{(2m)}(b))^{(\nu+1)/2m}((\phi(b+) + \phi(b-))/2m)$$
$$\exp(kh(b))\ \Gamma((\nu+1)/2m)$$

as $k \to \infty$.

Proof. If we write the integral as the integral from a to b plus the integral from b to c, and apply Corollary 1 to the first and Theorem 1 to the second integral, we arrive at the result directly.

As an example, we may find the asymptotic value of

$$\int_0^1 e^{-kx^4}\sin^6 x\,dx$$

This integral fulfills all the hypothesis of Theorem 1, since $\sin^6 x$ may be considered as $x^6\,\phi(x)$ with $\phi(0+) = 1$. Thus we have

$$\int_0^1 e^{-kx^8} \sin^6 x\,dx \sim \tfrac{1}{8} k^{-7/8}\,\Gamma(\tfrac{7}{8}) \qquad \text{as } k \to \infty.$$

Further, since

$$\left| \int_1^\infty e^{-kx^8} \sin^6 x\,dx \right| \leqslant e^{-(k-1)} \int_1^\infty e^{-x^8}\,dx, \; k \geqslant 1,$$

which is of smaller order than $k^{-7/8}$ as $k \to \infty$, we can say that

$$\int_0^\infty e^{-kx^8} \sin^6 x\,dx \sim \tfrac{1}{8} k^{-7/8}\,\Gamma(\tfrac{7}{8}) \qquad \text{as } k \to \infty.$$

Widder's results may be generalized in a different direction. The integral may be a double integral, and h and ϕ functions of two variables. The generalization is contained in the following theorem.

(1) *R is a measurable region of the $x - y$ plane, such that if*

$$S_\delta = \{(x, y) \mid (x-a)^2 + (y-b)^2 < \delta\},$$

then for some δ_0, $S_{\delta_0} \subset R$,

(2) $\phi(x, y) \in C^2(S_{\delta_0}) \cap L(R)$; $\phi_x = \phi_y = 0$, $\phi_{xy}{}^2 - \phi_{xx}\phi_{yy} < 0$, $\phi_{xx} < 0$, *at* (a, b); *for each* δ, $0 < \delta < \delta_0$, $\phi(x, y) \leqslant M(\delta) < \phi(a, b)$ *for* $(x, y) \in R - S_\delta$,

(3) $\psi(x, y) \in L(R)$, *$\psi(x, y)$ is continuous at (a, b) and* $\psi(a, b) \neq 0$, then

$$\int\int_R \exp\,(k\phi(x, y))\,\psi(x, y)\,dxdy$$

$$\sim 2\pi\,\psi(a, b)\exp\,(k\,\phi(a, b))/k(\phi_{xx}(a, b)\,\phi_{yy}(a, b) - \phi_{xy}{}^2(a, b))^{\frac{1}{2}}$$

as $k \to \infty$.

Proof. We may assume $\psi(a, b) > 0$. Let

$$x - a = \xi\cos\theta - \eta\sin\theta,\; y = \xi\sin\theta + \eta\cos\theta$$

where

$$\tan 2\theta = 2\phi_{xy}(a, b)/(\phi_{xx}(a, b) - \phi_{yy}(a, b));$$

let

$$\phi(x, y) = \Phi(\xi, \eta),\; \psi(x, y) = \Psi(\xi, \eta).$$

Then since

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = 1,$$

$$I_k = \int\int_R \exp \{k(\phi(x, y) - \phi(a, b))\} \psi(x, y)dxdy$$
$$= \int\int_R \exp (k(\Phi(\xi, \eta) - \Phi(0, 0)) \Psi(\xi, \eta)\, d\xi d\eta.$$

By 2, 3, and since $\Phi_{\xi\xi}(0, 0) < 0$, $\Phi_{\eta\eta}(0, 0) < 0$, $\Phi_{\xi\eta}(0, 0) = 0$, we can, for every ϵ, $0 < \epsilon < \min. (-\Phi_{\xi\xi}, -\Phi_{\eta\eta}, \Psi(0, 0))$, choose an $\alpha(\epsilon)$, $0 < \alpha < (\frac{1}{2}\delta_0)^{\frac{1}{2}}$, such that,

(i) $A_1 = \Phi_{\xi\xi}(0, 0) - \epsilon < \Phi_{\xi\xi}(\xi, \eta) < B_1 = \Phi_{\xi\xi}(0, 0) + \epsilon < 0,$

$$-\epsilon < \Phi_{\xi\eta}(\xi, \eta) < \epsilon,$$

$$A_2 = \Phi_{\eta\eta}(0, 0) - \epsilon < \Phi_{\eta\eta}(\xi, \eta) < B_2 = \Phi_{\eta\eta}(0, 0) + \epsilon < 0,$$

(ii) $0 < C_1 = \Psi(0, 0) - \epsilon < \Psi(\xi, \eta) < C_2 = \Psi(0, 0) + \epsilon,$

for all (ξ, η), $-\alpha \leqslant \xi \leqslant \alpha$, $-\alpha \leqslant \eta \leqslant \alpha$.

Let $R_\alpha = \{(x, y) | -\alpha \leqslant \xi \leqslant \alpha; -\alpha \leqslant \eta \leqslant \alpha\}$. Then

$$I_k = \left\{\int\int_{R_\alpha} + \int\int_{R-R_\alpha}\right\} \exp \{k(\phi(x, y) - \phi(a, b))\} = I_k' + I_k''.$$

Consider first I_k''. Since, for $(x, y) \in R - R_\alpha$, $(x-a)^2 + (y-b)^2 \geqslant \alpha^2$, we have, by (2),

$$0 \leqslant |I_k''| \leqslant \exp \{k(M(\alpha) - \phi(a, b))\} \int\int_R |\psi(x, y)|\, dxdy$$

so that $I_k'' = O(\omega^k)$ where $0 < \omega < 1$, since $\phi(a, b) > M(\alpha)$, and $\psi(x, y) \in L(R)$.

Now

$$I_k' = \int\int_{R_\alpha} \exp \{k(\phi(x, y) - \phi(a, b))\} \psi(x, y)\, dxdy$$
$$= \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \exp \{k(\Phi(\xi, \eta) - \Phi(0, 0))\} \Psi(\xi, \eta)\, d\xi d\eta$$
$$= \left\{\int_0^\alpha\int_0^\alpha + \int_{-\alpha}^0\int_0^\alpha + \int_0^\alpha\int_{-\alpha}^0 + \int_{-\alpha}^0\int_{-\alpha}^0\right\}$$
$$\exp \{k(\Phi(\xi, \eta) - \Phi(0, 0))\} \Psi(\xi, \eta)\, d\xi d\eta\cdot$$
$$= J_1 + J_2 + J_3 + J_4.$$

By (2) and Taylor's theorem,

$$\Phi(\xi, \eta) - \Phi(0, 0) = \tfrac{1}{2}(\Phi_{\xi\xi}(\theta\xi, \theta\eta)\, \xi^2 + 2\Phi_{\xi\eta}(\theta\xi, \theta\eta)\, \xi\eta + \Phi_{\eta\eta}(\theta\xi, \theta\eta)\, \eta^2),$$
$$0 < \theta < 1.$$

Consider J_1. By (i) and (ii),

$$C_1 \int_0^\alpha\int_0^\alpha \exp (\tfrac{1}{2}k(A_1\xi^2 - 2\epsilon\xi\eta + A_2\eta^2))\, d\xi d\eta < J_1$$
$$< C_2 \int_0^\alpha\int_0^\alpha \exp (\tfrac{1}{2}k(B_1\xi^2 + 2\epsilon\xi\eta + B_2\eta^2))\, d\xi d\eta.$$

In the left hand side, let $-u^2 = \frac{1}{2}kA_1\xi^2$, $-v^2 = \frac{1}{2}kA_2\eta^2$, and in the right hand side, let $-u^2 = \frac{1}{2}kB_1\xi^2$, $-v^2 = \frac{1}{2}kB_2\eta^2$. Then the inequalities become

$$(2C_1/k(A_1A_2)^{\frac{1}{2}}) \int_0^{k^{1/2}D_1} \int_0^{k^{1/2}D_2} \exp(-(u^2 + 2\epsilon(A_1A_2)^{-\frac{1}{2}} uv + v^2))\, dudv$$

$$< J_1 < (2C_2/k(B_1B_2)^{\frac{1}{2}}) \int_0^{k^{1/2}D_3} \int_0^{k^{1/2}D_4}$$

$$\exp(-(u^2 - 2\epsilon(B_1B_2)^{-\frac{1}{2}}uv + v^2))\, dudv,$$

where $D_i = \alpha(-\frac{1}{2}A_i)^{\frac{1}{2}}$, $(i = 1, 2)$ and $D_i = \alpha(-\frac{1}{2}B_i)^{\frac{1}{2}}$, $(i = 3, 4)$.
Hence

$$(2C_1/(A_1A_2)^{\frac{1}{2}}) \int_0^{\infty} \int_0^{\infty} \exp(-(u^2 + 2\epsilon(A_1A_2)^{-\frac{1}{2}}uv + v^2))\, dudv$$

$$\leqslant \underline{\overline{\lim}}\, kJ_1 \leqslant (2C_2/(B_1B_2)^{\frac{1}{2}}) \int_0^{\infty} \int_0^{\infty}$$

$$\exp(-(u^2 + 2\epsilon(B_1B_2)^{-\frac{1}{2}}\, uv + v^2))\, dudv$$

and thus, since ϵ is arbitrary, and the integrals in question converge uniformly in ϵ for $0 \leqslant \epsilon \leqslant \epsilon_0 < \frac{1}{2}(B_1B_2)^{\frac{1}{2}}$, we have

$$\lim_{k\to\infty} kJ_1 = (2\Psi(0,0)/(\Phi_{\xi\xi}(0,0)\Phi_{\eta\eta}(0,0))^{\frac{1}{2}} \int_0^{\infty} \int_0^{\infty} e^{-(u^2+v^2)}\, dudv$$

$$= \pi\, \psi(a,b)/2(\phi_{xx}(a,b)\, \phi_{yy}(a,b) - \phi_{xy}^{\ 2}(a,b))^{\frac{1}{2}}.$$

Similarly,

$$\lim_{k\to\infty} kJ_2 = \lim_{k\to\infty} kJ_3 = \lim_{k\to\infty} kJ_4 = \pi\, \psi(a,b)/2(\phi_{xx}(a,b)\, \phi_{yy}(a,b)$$

$$- \phi_{xy}^{\ 2}(a,b))^{\frac{1}{2}}.$$

Hence

$$\lim_{k\to\infty} kI_k = \lim_{k\to\infty} kI_k' = \lim_{k\to\infty} k(J_1 + J_2 + J_3 + J_4)$$

$$= 2\pi\, \psi(a,b)/(\phi_{xx}(a,b)\, \phi_{yy}(a,b) - \phi_{xy}^{\ 2}(a,b))^{\frac{1}{2}}.$$

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TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA
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SECTION THREE

A Remark on Curves of Order n in n-space

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1. Let R_n denote real projective n-space. An arc A [a curve C] is the continuous image of an interval [of a circle] in R_n. The order of A is the upper limit of the number of points which A may have in common with any [linear] $(n-1)$-space. We exclude the case that A degenerates into a single point. Then the order of A is not less than n. Let A^n [c^n] denote an arc [a curve] of order n in R_n. It necessarily is a Jordan arc [a Jordan curve].

Let $s, s', \ldots$ denote points on the arc A. We call the point s differentiable if it possesses osculating subspaces $L_p{}^n(s)$ of every dimension $[p = -1, 0, 1, \ldots, n]$. Let $L_{-1}{}^n(s)$ be empty space. Suppose we have defined $L_p{}^n(s)$ and postulated its existence. Then we require that the $(p+1)$-spaces through $L_p{}^n(s)$ and a point s' converging to s have a unique limit space. We call it the osculating $(p+1)$-space $L_{p+1}{}^n(s)$. Thus $L_0{}^n(s)$ is the point s itself and $L_n{}^n(s)$ is the whole R_n. We call A differentiable if each of its points is.

It is well known [1] that the end-points of an arc A^n are differentiable. Thus, the interior points of A^n are "one-sidedly differentiable." This implies that A^n is rectifiable and that it is differentiable everywhere with not more than a countable number of points as exceptions.

2. In a recent paper [4], Schoenberg proved with analytic methods that every C^n can be uniformly approximated by analytic C^n's if n is even. He also proved for odd n's that every A^n can be uniformly approximated by arcs of analytic C^n's with the same end-points. In this note, a geometrical proof of the following weaker theorem will be outlined:

Every C^n can be uniformly approximated by differentiable C^n's.

Here n may be any integer greater than one. Our proof is based on [2] and [3]. The author should like to acknowledge the stimulus he received from Dr. Schoenberg and Dr. Motzkin.

3. Given an $(n-1)$-space E and an interior point s of an arc A^n. Let $\bar{A}^n$ be a small closed subarc of A^n which contains s in its interior and which meets E nowhere else. We then call s a point of support

[of intersection] with respect to E if the end-points of $\bar{A}^n$ lie on the same side of E [are separated by E]. The following lemma is quoted from [3]: Suppose s is differentiable and E contains $L_p{}^n(s)$ but not $L_{p+1}{}^n(s)$. If p is odd [even], then s is a point of support [of intersection] with respect to E.

4. Let s and s' be two differentiable points on a curve C^n; $s \neq s'$. They divide C^n into two arcs A^n and A'^n.

The osculating spaces $L_p{}^n(s)$ and $L_{n-p-2}{}^n(s')$ are readily seen to span an $(n-1)$-space which does not meet C^n outside of s and s' and which does not contain $L_{p+1}{}^n(s)$ or $L_{n-p-1}{}^n(s')$ $[-1 \leqslant p \leqslant n-1]$. Altogether there are $n+1$ subspaces of this kind. We verify by induction that they subdivide R_n into 2^n simplexes. From the above, A^n $[A'^n]$ lies entirely in one of these simplexes. The lemma quoted in §3 implies that the simplex S that A'^n lies in is determined by A^n.

Let A''^n be a third arc of order n with the same end-points s and s'. Suppose $A^n \cup A''^n$ is differentiable at s and at s' and shares with C^n the property that the $(n-1)$-space through $L_p{}^n(s)$ and $L_{n-p-2}{}^n(s')$ supports or intersects $A^n \cup A''^n$ at s if p is even respectively odd $[p = 0, 1, 2, \ldots, n-1]$. Then, from [2] and [3], $A^n \cup A''^n$ is a curve of order n. From the above, A''^n will also lie in S. There are differentiable arcs A''^n with the required properties. As a matter of fact, A''^n can be an arc of a rational curve of degree n.

5. Given the curve C^n of order n. We introduce an elliptic metric into R_n and decompose C^n by means of differentiable points into sufficiently small subarcs A'^n. If our decomposition is fine enough, then each of the simplexes S determined by the A'^n will be arbitrarily small. Replacing consecutively each A'^n by a differentiable A''^n, we obtain a curve of order n which is differentiable everywhere and which lies in the union of the simplexes S. This proves our theorem.

Schoenberg's theorem could be proved in this fashion if the following conjecture is true: To any finite set of differentiable points on a C^n there exists an analytic curve of order n through them which has at each of them the same osculating spaces that C^n has.

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SECTION THREE

The Application of Fourier Transforms in Physical Problems

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THE application of Fourier transforms, and integral transforms in general, is of quite modern interest for it is still evolving. Its study exposes in a fundamental way the interactions of mathematics and physics. In the main, books about it suffer from one or other of two faults. Mathematical interest is usually confined to the theory of some special transform such as Mellin or Laplace, and the applications discussed are often trivial, illustrating basic points of theory it is true, but nevertheless falling short as a guide to the detailed analytical technique needed to handle real problems. This afternoon it would be out of place to venture on mathematical elaboration that properly belongs to sustained study of the subject. It is appropriate, however, to recall in mathematical form the basis of integral transforms.

Given a function $K(\alpha, x)$ of two variables α and x, and the function $f(x)$, the function $F(\alpha)$ defined by

$$F(\alpha) = \int_0^\infty f(x)\, K(\alpha, x)\, dx$$

is, when the integral is convergent, the integral transform of the function $f(x)$ by the kernel $K(\alpha, x)$.

In the Laplace transform $K(\alpha, x) = e^{-\alpha x}$
In the Mellin transform $K(\alpha, x) = x^{\alpha - 1}$
In the Fourier cosine transform $K(\alpha, x) = \cos \alpha x$
In the Fourier sine transform $K(\alpha, x) = \sin \alpha x$
In the Hankel transform of order ν $K(\alpha, x) = (\alpha x)^{\frac{1}{2}} J_\nu(\alpha x)$

In each case there is an appropriate inversion formula by which $f(x)$ may be deduced from $F(\alpha)$ [7].

The Fourier transform of $f(x)$ is

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx \tag{1}$$

and the inversion relation is

$$(2) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{-i\alpha x}\, d\alpha$$

$f(x)$ must satisfy Dirichlet's conditions and

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

must converge.

It is forty years since Sommerfeld applied Fourier transformation to treat the propagation of a signal in a dispersive medium. The problem arose from the objection offered to Einstein's Special Relativity theory—an objection supported by citing examples of media in which, since $dV/d\lambda$ is negative, the group velocity exceeds c, and put forward in the belief that the group velocity is necessarily equivalent to the velocity of energy propagation. The particular problem considered by Sommerfeld [8] is that of a signal which originates at $z = 0$, at instant $t = 0$, and continues indefinitely thereafter as a harmonic oscillation of frequency ω_0 (radians/s).

That the Fourier integral of a non-terminating harmonic wave does not converge may be seen from equation (1). The difficulty may be circumvented by deforming the path of integration into the complex domain of the variable ω_0. For example, for

$$f(t) = \begin{cases} \dfrac{1}{2\pi} e^{-i\omega_0 t} & t > 0 \\ 0 & t > 0, \end{cases}$$

$$(3) \qquad \bar{f}(\omega) = \frac{1}{2\pi}\int_{0}^{\infty} e^{i(\omega-\omega_0)t}\, dt = \frac{1}{2\pi i\,(\omega_0 - \omega)}$$

provided Im $(\omega - \omega_0) > 0$; the integral then converges. Inversely,

$$(4) \qquad f(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{2\pi(\omega_0 - \omega)}\, dt.$$

This integral is evaluated by the theory of residues, the contour being closed by a semicircle at infinity on which, by Jordan's lemma, the contribution to the contour integral is zero. When $t > 0$, the path of integration lies in the negative half-plane at infinity thus enclosing the pole $\omega = \omega_0$. Whereas when $t < 0$ it lies above the real axis and excludes the pole, as shown in Figure 1. This gives the essential discontinuity in the form $f(t)$.

To obtain the Fourier transform of

$$e^{-i\omega_0 t} \qquad (-\infty < t < \infty)$$

we evidently have to combine

$$\bar{f}_+(\omega) = \frac{1}{2\pi}\int_0^\infty e^{i(\omega-\omega_0)t}\,dt \quad \text{and} \quad \bar{f}_-(\omega) = \frac{1}{2\pi}\int_{-\infty}^0 e^{i(\omega-\omega_0)t}\,dt$$

requiring $\mathrm{Im}(\omega - \omega_0) > 0$ for the first and $\mathrm{Im}(\omega - \omega_0) < 0$ for the second. It is in this sense that we write

$$(5) \quad \delta(\omega - \omega_0) = \delta_+(\omega - \omega_0) + \delta_-(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t}\,dt$$

and regard $\delta(\omega - \omega_0)$ as the transform of

$$\frac{1}{2\pi}e^{-i\omega_0 t}.$$

δ has been called the Dirac δ-function. Schwartz [6] calls it the Dirac measure. It is not a function. I shall return to this topic later in the lecture.

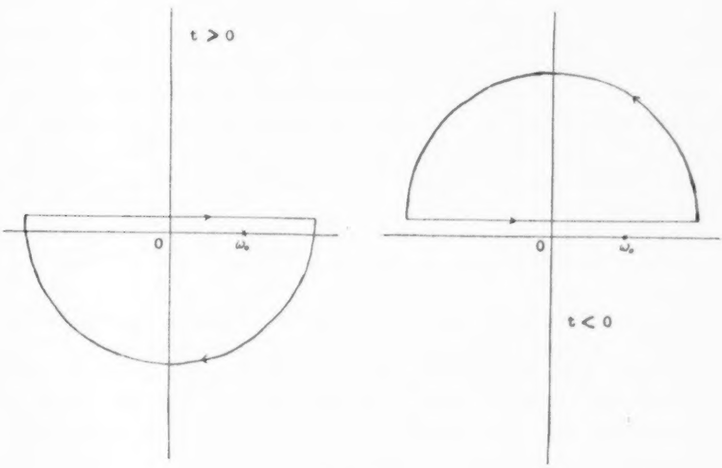

FIGURE 1

The broad use of integral transforms to solve differential equations has developed from Heaviside's operator calculus, devised by him to solve transient problems in physics and electrical engineering. The important core of Heaviside's method is that it corresponds to the possibility of computation with numbers instead of analysis because it starts with what is given, and evolves the answer explicitly by direct operations on the data.

If one wished to characterize the role of Fourier transformation in the theory of physical systems, it may, I think, be fairly stated that Fourier transformation is the natural extension of Fourier series to unbounded spaces permitting the representation of discontinuous distributions in space and time by methods conformable to analysis. Whether discontinuity is a physical necessity does not really matter: it is mathematically simpler than the representation of the actual rapid transition according to any special law that might be physically appropriate, but which, nevertheless, is judged irrelevant to the main idealized purpose of the calculation. This use of discontinuity in representing nature is quite analogous to the use of impulses in dynamics.

It has already been pointed out how the switching on of an oscillation is managed by treating Fourier transformations in the complex domain of the transform variable: a great deal of the subtlety of calculation emerges from this analytical way of representing discontinuous distributions. Since the physicist expects to see in the mathematical development forms of which he recognizes the physical relevance, he will tend to by-pass whatever he thinks he may dismiss as mathematical ritual. This is in some respects very unfortunate, because he is often thereby deprived of the stimulus to appreciate the delicacy of the conceptions being treated and so may not learn to use the very sharp tools of the mathematician.

Let us now make a list of topics in the study of which the application of Fourier transformation has played an important role for theoretical physics.

(a) In the theory of vibrations, Fourier series are the main tool, integral transforms apply usually only to the time dependence.

(b) The diffusion equation and that of heat conduction we might regard as simplified versions of the transport equation and indeed as representing the whole field of continuous linear stochastic theory.

(c) In elasticity stress systems due to discontinuous pressure distributions have been treated.

(d) Hydrodynamical problems involving discontinuous boundary conditions.

(e) Electromagnetic theory and electromechanics have been the field of many successful applications.

(f) Fourier transformation is peculiarly suited for the study of wave equations in general and especially to provide powerful methods for handling singularities and discontinuities.

(g) In modern physics one can point to the use of wave functions in momentum space to treat scattering problems, the use of Mellin and

Laplace transforms in the cascade theories for cosmic rays, and the wide use of Fourier transformation in quantum electrodynamics and wave field theory.

For the mathematician, however, the application of Fourier transformation to physical problems consists of its use to solve (i) partial differential equations and systems of them, (ii) integral equations and systems thereof and in general, (iii) integro-differential equation systems. Of course, Fourier transformation has its role in mathematics itself, and it is an important tool in statistics.

This is an impressive list and naturally not much of it can be exhibited in this lecture. I propose to pick out a few topics to illustrate what has been going on.

The theory of the slowing-down and diffusion of neutrons has received much attention because of its importance for understanding nuclear reactors. We shall consider a very small part of the theory of neutron diffusion under extremely rigorous (indeed unrealistic) physical assumptions designed to simplify the problem [5]. We begin with the transport equation by which the rate of change of the neutron density following the motion is accounted for by capture, scattering and the presence of neutron sources:

$$\frac{d\psi(\mathbf{r},\beta,t)}{dt} = -P(\beta)\psi(\mathbf{r},\beta,t) + \int P_s(\beta')f(\beta,\beta')\,d\beta'\psi(\mathbf{r},\beta',t) + q(\mathbf{r},\beta,t)$$

with

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \operatorname{div}(\mathbf{v}\,\psi). \tag{6}$$

The functions ψ and q denote respectively the density of neutrons and of neutron sources both per unit volume at the space position $\mathbf{r}$ at time t, and per unit range of the symbol β that stands for all other parameters required to specify a neutron as, for instance, its velocity $\mathbf{v}$ or direction of motion $\mathbf{S}$. The probabilities per unit time that a neutron is scattered or captured from the range $d\beta$ of β are respectively $P_s(\beta)$ or $P_c(\beta)$. Their sum is $P(\beta)$, the total probability of collision. The relative probability that a neutron originally with parameters β' will be scattered into the range $(\beta', \beta' + d\beta')$ is denoted by $f(\beta', \beta)\,d\beta'$.

The simplifications we shall introduce are:

(a) scattering without change of energy, all neutrons having the same speed v which will be taken as unit;

(b) scattering and sources isotropic;

(c) stationary state.

The mean free path (l) being taken as the unit of length, the unit of time is l/v. Then $P = 1$, $P_c = \alpha$, and $P_s = 1 - \alpha$ and we have in place of (6)

$$\text{(7)} \qquad \mathbf{S} \,.\, \text{grad}\, \psi(\mathbf{r}, \mathbf{S}) + \psi(\mathbf{r}, \mathbf{S}) = \frac{1-\alpha}{4\pi}\psi_0(\mathbf{r}) + \frac{1}{4\pi} q_0(\mathbf{r})$$

where

$$\psi_0(\mathbf{r}) = \int \psi(\mathbf{r}, \mathbf{S}')\, d\mathbf{S}'$$

is the neutron density at $\mathbf{r}$, and $d\mathbf{S}$ is the element of solid angle.

The integro-differential equation (7) to determine ψ is equivalent to an algebraic equation on Fourier transformation, viz.

$$\text{(8)} \qquad - i\mathbf{k} \,.\, \mathbf{S}\, \phi(\mathbf{k}, \mathbf{S}) + \phi(\mathbf{k} \,.\, \mathbf{S}) = \frac{1-\alpha}{4\pi}\phi_0(\mathbf{k}) + \frac{1}{4\pi}\lambda_0(\mathbf{k})$$

where

$$\phi(\mathbf{k}, \mathbf{S}) = \int \psi(\mathbf{r}, \mathbf{S}) \exp\,(i\mathbf{k} \,.\, \mathbf{r})\, r^2\, dr \sin\theta\, d\theta\, d\phi$$

and $\phi_0(\mathbf{k})$ and $\lambda_0(\mathbf{k})$ are the corresponding integrals for $\psi_0(r)$ and $q_0(r)$. The differential operator $\mathbf{S}$.grad becomes the scalar product.

Now rearrange equation (8) thus

$$\phi(\mathbf{k}, \mathbf{S}) = \frac{1}{4\pi(1 - i\mathbf{k} \,.\, \mathbf{S})}\{(1-\alpha)\,\phi_0(\mathbf{k}) + \lambda_0(\mathbf{k})\}.$$

Integrate over all directions $\mathbf{S}$ to obtain

$$\text{(9)} \qquad \phi_0(\mathbf{k}) = \{(1-\alpha)\,\phi_0(\mathbf{k}) + \lambda_0(\mathbf{k})\}\int \frac{d\mathbf{S}/4\pi}{1 - i\mathbf{k} \,.\, \mathbf{S}}.$$

Since the integral is equal to $k^{-1}\tan^{-1} k$, we may obtain the transform $\phi_0(\mathbf{k})$ of the neutron density directly in terms of the transform $\lambda_0(\mathbf{k})$ of the density of sources:

$$\text{(10)} \qquad \phi_0(\mathbf{k}) = \frac{\tan^{-1}k}{k - (1-\alpha)\tan^{-1}k}\lambda_0(\mathbf{k}),$$

which has the form

$$\text{(11)} \qquad \phi_0(\mathbf{k}) = g(k)\,\lambda_0(\mathbf{k}).$$

Because the operations multiplication and convolution correspond on transformation and since $g(k)$ is the Fourier transform of $e^{-r}/4\pi r^2$, the equation (11) reads after inversion

$$\text{(12)} \qquad \psi_0(\mathbf{r}) = \int \frac{e^{-R}}{4\pi R^2} q_0(\mathbf{r}')\, d\mathbf{r}',$$

where $R = |\mathbf{r} - \mathbf{r}'|$, showing how the source element at $\mathbf{r}'$ contributes to the neutron density at $\mathbf{r}$. The symbol $d\mathbf{r}'$ denotes the volume element in the space of $\mathbf{r}'$.

As a second problem we shall consider the decay of resonance radiation by spontaneous emission to determine the shape of the resonance line. The physical system consists of one atom initially excited and the radiation field initially devoid of photons. It is represented by a state vector ψ which is a function of the state of the atom and the numbers N_σ of the photons in the states σ. Since we are dealing with resonance radiation we restrict ourselves to the excited state A and the ground state B of the atom and introduce the two relevant basis vectors

$$a(t) \equiv \psi(A, 0_\sigma) \text{ and } b(\nu, t) \equiv \psi(B, 1_\sigma).$$

The fundamental quantum equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi \tag{13}$$

is expressed by the pair

$$\begin{aligned} -i\frac{\partial a}{\partial t} &= \int_0^\infty W(\nu)\, b(\nu, t)\, S(\nu)\, d\nu \\ -i\frac{\partial b}{\partial t} &= W^* a + 2\pi(\nu - \nu_1)b \end{aligned} \tag{14}$$

where $\hbar W(\nu)$ is the interaction operator of the Hamiltonian ν being the photon frequency, and

$$h\nu_1 = E_A - E_B, \qquad S(\nu) = \frac{8\pi^2\nu^2 V}{c^3},$$

and V is the volume of the hohlraum. Equation system (14) is to be solved subject to $a(0) = 1$, $b(\nu, 0) = 0$. On Fourier transformation with respect to t, the transform variable being z (and W^* the complex conjugate of W) we obtain

$$-z\bar{a}(z) = \int_0^\infty W(\nu)\, \bar{b}(\nu, z)\, S(\nu)\, d\nu + a(0) \tag{15a}$$

$$-z\bar{b}(\nu, z) = 2\pi(\nu - \nu_1)\, \bar{b}(\nu, z) + W^*\bar{a}(z). \tag{15b}$$

Solve (15b) for $\bar{b}(\nu, z)$, substitute in (a) and find $a(z)$, which in turn is used to give $\bar{b}(\nu, z)$. On inversion we obtain an integral formula for $b(\nu, t)$, a formal solution from which the shape of the resonance line has been worked out [4].

What I have stated regarding the special connexion of Fourier transformation with discontinuity applies in a peculiarly significant way to the treatment of waves.

For Laplace's equation $\Delta U = 0$ and for the iterated Laplace's equations $\Delta^n U = 0$, which are of elliptic type, there are no solutions

other than the analytic functions. However, for a hyperbolic equation such as

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 0$$

the general solution will be

$$U = A(x + y) + B(x - y)$$

where A and B *may* be continuous functions, not necessarily differentiable, discontinuous functions and even not functions at all in the sense of classical analysis. The introduction of such discontinuous solutions in the theory of partial differential equations is of great importance to physics. It is remarkable that substantially the unorthodoxy initiated by Heaviside is still continuing. The latest adjustment of mathematics to physical intuition is exhibited in the noteworthy Theory of Distributions due to Schwartz. His book on this subject appeared in 1950 and members of this audience who attended the meeting of the Canadian Mathematical Congress at Vancouver in 1949 will recall the excellent series of lectures he delivered on distribution theory. This theory has found a proper mathematical basis for handling discontinuous solutions for partial differential equations and of escaping from the mathematical category of functions to treat Fourier transformation.

For the physicist a constant harmonic wave of infinite duration has an infinitely sharp line spectrum. It is represented by a δ-measure. The inverse relation surely requires the Dirac measure δ to have a Fourier transform. Here physical intuition has kept ahead of mathematics. For

$$\mathcal{F}\delta(\omega - \omega_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{-i\omega t}\, d\omega = \frac{1}{2\pi} e^{-i\omega_0 t}$$

Now this formal equation we have written down does not fall under the terms of classical analysis, in fact in Lebesgue's theory of integration the integral is zero. Accordingly this formalism can be kept "on the rails" only by physical understanding, whereas the value of mathematics is that logically correct answers are assured by obeying the rules of calculation. Schwartz's work seems likely to improve the situation but there will be a period when notation will be muddled—a matter of serious practical effect in modern theoretical physics.

It is ten years since Heisenberg [3] invented the S-matrix that plays so important a part in quantum electrodynamics and the theory of wave fields. In his papers he represented the asymptotic wave functions, representing for example emergent particles from a reaction, by Fourier

transforms. For instance $\delta_+(k_0 - k)$ associated with time dependence $\exp(-ik_0ct)$ represents an outgoing spherical wave as may be seen as follows.

$$\mathscr{F}\delta_+(k_0 - k) = \frac{1}{(2\pi)^3}\int_0^\infty\int_{-1}^1\int_0^{2\pi}\frac{e^{-ikr\zeta}}{2\pi i\,(k_0 - k)}\,d\phi\,d\zeta\,k^2dk$$

$$(Im(k_0 - k) > 0)$$

$$= \frac{1}{(2\pi)^3}\frac{1}{r}\int_0^\infty\frac{k(e^{ikr} - e^{-ikr})}{(k_0 - k)}\,dk.$$

We assume r large: since k_0 is real Im $k < 0$, and hence the main contribution to the integral is

$$\frac{1}{(2\pi)^3 r}\int_0^\infty\frac{k\,e^{ikr}}{k_0 - k}\,dk$$

which in turn comes from the vicinity of k_0, the result being

$$\frac{-ik_0\,e^{ik_0r}}{4\pi^2r}.$$

In the quantum electrodynamics of Schwinger, Feynman, Dyson *et al.*, one has to deal with the singular wave functions associated with sources and sinks to represent the creation and annihilation of particles. Quite apart from the question of relativistic invariance, the essential mathematical difficulties are concerned with the fact that the elementary solutions of the various wave equations (due to point singularities) are properly speaking not functions at all. Nevertheless one writes down expressions for their Fourier transforms.

Fourier transformation enters as the obvious means for dealing with the convolution integrals that appear in the calculation of the wave functions representing the quantum stochastic process [1]. For example, a particle starts from A travels to B where it is scattered, what is the probability amplitude to reach C from A after such a single scattering process? It has the form

$$\int g(\mathbf{x} - \mathbf{x}')\,f(\mathbf{x}')\,d\mathbf{x}'$$

and its Fourier transform is $G(k)F(k)$ which is a straightforward product. The transforms are all singular and it is in the handling of them by conventional means that some of the difficult divergencies in the theory arise. In a recent paper, Güttinger [2] has shown how Schwartz's Distribution Analysis automatically leaves a place for mathematical operations equivalent in effect to renormalization, regularization, or cut-off procedures: it does this quite naturally and brings to light that the theory as at present conceived is physically

incomplete. There is a good expectation that distribution analysis may remove some of the mathematical difficulties of quantum field theory especially as applied to mesons.

Thus we view today an interesting prospect. New mathematical tools have become available to the theoretical physicist, but perhaps even more promising is the realization that mathematicians have been too ready in the past to impose on physics the unnecessarily restrictive categories of their own inventions. It is significant that the application of Fourier transforms has been the source of these developments that illuminate the role of mathematics in relation to physics.

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